An extension of Nunokawa lemma (On Schwarzian Derivatives and Its Applications)

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Abstract

Let $\mathcal{H}[a_0, n]$ be the class of functions $p(z) = a_0 + a_n z^n + \cdots$ which are analytic in the open unit disk $\mathbb{U}$. For functions $f(z)$ which are analytic in $\mathbb{U}$ with $f(0) = 1$, M. Nunokawa (Proc. Japan Acad., Ser. A **68** (1992), 152–153) have shown some theorems. The object of the present paper is to discuss Nunokawa lemma for the class $\mathcal{H}[a_0, n]$.

1 Introduction

Let $\mathcal{H}[a_0, n]$ denote the class of functions $p(z)$ of the form

$$p(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for some $a_0 \in \mathbb{C}$ and a positive integer n.

The basic tool in proving our results is the following lemma due to S. S. Miller and P. T. Mocanu [1] (also [2]).

Lemma 1. *Let the function $w(z)$ definded by*

$$w(z) = a_n z^n + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots \qquad (n = 1, 2, 3, \cdots)$$

be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in \mathbb{U}$, then there exists a real number $m \geqq n$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = m.$$

2 Main result

Applying Lemma 1, we derive the following result.

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Theorem 1. *Let $p(z) \in \mathcal{H}[a_0, n]$ for some real $a_0 > 0$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\operatorname{Re}(p(z)) > 0 \quad for \quad |z| < |z_0|$$

and $p(z_0) = \beta i$ is a pure imagenary number for some real $\beta \neq 0$.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = il$$

where

$$l \geqq \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \geqq n$$

if $\beta > 0$ and

$$l \leqq \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \leqq -n$$

if $\beta < 0$.

Proof. Let us put

$$w(z) = \frac{a_0 - p(z)}{a_0 + p(z)} = c_n z^n + c_{n+1} z^{n+1} + c_{n+2} z^{n+2} + \cdots \qquad (z \in \mathbb{U}).$$

Then, we have that $w(z)$ is analytic in $|z| < |z_0|$, $w(0) = 0$, $|w(z)| < 1$ for $|z| < |z_0|$ and

$$|w(z_0)| = \left| \frac{a_0^2 - \beta^2 - 2a_0\beta i}{a_0^2 + \beta^2} \right| = 1.$$

From Lemma 1, we obtain

$$\frac{z_0 w'(z_0)}{w(z_0)} = \frac{-2a_0 z_0 p'(z_0)}{a_0^2 - \{p(z_0)\}^2} = \frac{-2a_0 z_0 p'(z_0)}{a_0^2 + \beta^2} = m \qquad (m \geqq n).$$

This shows that

$$z_0 p'(z_0) = -\frac{m}{2}\left(a_0 + \frac{\beta^2}{a_0}\right) \qquad (m \geqq n).$$

From the fact that $z_0 p'(z_0)$ is a real number and $p(z_0)$ is a pure imaginary number, we can put

$$\frac{z_0 p'(z_0)}{p(z_0)} = il$$

where l is a real number.

For the case $\beta > 0$, we have

$$\begin{aligned} l &= \operatorname{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= \operatorname{Im}\left(-z_0 p'(z_0)\frac{1}{\beta}i\right) \\ &= \frac{m}{2}\left(a_0 + \frac{\beta^2}{a_0}\right) \\ &\geqq \frac{n}{2}\left(a_0 + \frac{\beta^2}{a_0}\right)\frac{1}{\beta} \\ &= \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \geqq n \end{aligned}$$

and for the case $\beta < 0$, we get

$$\begin{aligned} l &= \operatorname{Im}\left(\frac{z_0 p'(z_0)}{p(z_0)}\right) \\ &= \operatorname{Im}\left(-z_0 p'(z_0)\frac{1}{\beta}i\right) \\ &= \frac{m}{2}\left(a_0 + \frac{\beta^2}{a_0}\right) \\ &\leqq \frac{n}{2}\left(a_0 + \frac{\beta^2}{a_0}\right)\frac{1}{\beta} \\ &= \frac{n}{2}\left(\frac{a_0}{\beta} + \frac{\beta}{a_0}\right) \leqq -n. \end{aligned}$$

This completes our proof. □

Putting $a_0 = 1$ in Theorem 1, we have Corollary 1.

Corollary 1. *Let $p(z) \in \mathcal{H}[1, n]$ and suppose that there exists a point $z_0 \in \mathbb{U}$ such that*

$$\operatorname{Re}(p(z)) > 0 \quad \textit{for} \quad |z| < |z_0|,$$

$\operatorname{Re}(p(z_0)) = 0$ *and* $p(z_0) \neq 0$.

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = il$$

where l is a real and $|l| \geqq n$.

References

[1] S. S. Miller and P. T. Mocanu, *Second-order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65**(1978), 289-305.

[2] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, Theory and Applications. Monographs and Textbooks in Pure and Applied Mathematics, 225. Marcel Dekker, Inc., New York, 2000.

[3] M. Nunokawa, *On propeties of non-Carathéodory functions*, Proc. Japan Acad., Ser. A **68** (1992), 152–153.

[4] M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci. **69**(1993), 234–237.

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